

DIRECT PRODUCTS IN PROJECTIVE SEGRE CODES

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ABSTRACT. Let $K = \mathbb{F}_q$ be a finite field. We introduce a family of projective Reed-Muller-type codes called *projective Segre codes*. Using commutative algebra and linear algebra methods, we study their basic parameters and show that they are direct products of projective Reed-Muller-type codes. As a consequence we recover some results on projective Reed-Muller-type codes over the Segre variety and over projective tori.

1. INTRODUCTION

Reed-Muller-type evaluation codes have been extensively studied using commutative algebra methods (e.g., Hilbert functions, resolutions, Gröbner bases); see [3, 10, 27] and the references therein. In this paper we use these methods—together with linear algebra techniques—to study projective Segre codes over finite fields.

Let K be an arbitrary field, let a_1, a_2 be two positive integers, let $\mathbb{P}^{a_1-1}, \mathbb{P}^{a_2-1}$ be projective spaces over K , and let $K[\mathbf{x}] = K[x_1, \dots, x_{a_1}]$, $K[\mathbf{y}] = K[y_1, \dots, y_{a_2}]$, $K[\mathbf{t}] = K[t_{1,1}, \dots, t_{a_1,a_2}]$ be polynomial rings with the standard grading. If $d \in \mathbb{N}$, let $K[\mathbf{t}]_d$ denote the set of homogeneous polynomials of total degree d in $K[\mathbf{t}]$, together with the zero polynomial. Thus $K[\mathbf{t}]_d$ is a K -linear space and $K[\mathbf{t}] = \bigoplus_{d=0}^{\infty} K[\mathbf{t}]_d$. In this grading each $t_{i,j}$ is homogeneous of degree one.

Given $\mathbb{X}_i \subset \mathbb{P}^{a_i-1}$, $i = 1, 2$, denote by $I(\mathbb{X}_1)$ (resp. $I(\mathbb{X}_2)$) the *vanishing ideal* of \mathbb{X}_1 (resp. \mathbb{X}_2) generated by the homogeneous polynomials of $K[\mathbf{x}]$ (resp. $K[\mathbf{y}]$) that vanish at all points of \mathbb{X}_1 (resp. \mathbb{X}_2). The *Segre embedding* is given by

$$\begin{aligned} \psi: \mathbb{P}^{a_1-1} \times \mathbb{P}^{a_2-1} &\rightarrow \mathbb{P}^{a_1 a_2 - 1} \\ [(\alpha_1, \dots, \alpha_{a_1}), (\beta_1, \dots, \beta_{a_2})] &\rightarrow [(\alpha_i \beta_j)], \end{aligned}$$

where $[(\alpha_i \beta_j)] := [(\alpha_1 \beta_1, \alpha_1 \beta_2, \dots, \alpha_1 \beta_{a_2}, \dots, \alpha_{a_1} \beta_1, \alpha_{a_1} \beta_2, \dots, \alpha_{a_1} \beta_{a_2})]$. The map ψ is well-defined and injective [20, p. 13]. The image of $\mathbb{X}_1 \times \mathbb{X}_2$ under the map ψ , denoted by \mathbb{X} , is called the *Segre product* of \mathbb{X}_1 and \mathbb{X}_2 . The vanishing ideal $I(\mathbb{X})$ of \mathbb{X} is a graded ideal of $K[\mathbf{t}]$, where the $t_{i,j}$ variables are ordered as $t_{1,1}, \dots, t_{1,a_2}, \dots, t_{a_1,1}, \dots, t_{a_1,a_2}$. The Segre embedding is used in algebraic geometry, among other applications, to show that the product of projective varieties is again a projective variety, see [19, Lecture 2]. If $\mathbb{X}_i = \mathbb{P}^{a_i-1}$ for $i = 1, 2$, the set \mathbb{X} is a projective variety and is called a *Segre variety* [19, p. 25]. The Segre embedding is used in coding theory, among other applications, to study the generalized Hamming weights of some product codes; see [29] and the references therein.

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The contents of this paper are as follows. Let $K = \mathbb{F}_q$ be a finite field. In Section 2 we recall two results about the basic parameters and the second generalized Hamming weight of direct product codes (see Theorems 2.1 and 2.2). Then we introduce the family of projective Reed-Muller-type codes, examine their basic parameters, and explain the relation between Hilbert functions and projective Reed-Muller-type codes (see Proposition 2.7). For an arbitrary field K we show that $K[\mathbf{t}]/I(\mathbb{X})$ is the Segre product of $K[\mathbf{x}]/I(\mathbb{X}_1)$ and $K[\mathbf{y}]/I(\mathbb{X}_2)$ (see Definition 2.8 and Theorem 2.10). The Segre product is a subalgebra of

$$(K[\mathbf{x}]/I(\mathbb{X}_1)) \otimes_K (K[\mathbf{y}]/I(\mathbb{X}_2)),$$

the tensor product algebra. Segre products have been studied by many authors; see [9, 18, 21] and the references therein. We give full proofs of two results for which we could not find a reference with the corresponding proof (see Lemma 2.3 and Theorem 2.10). Apart from this all results of this section are well known.

If $K = \mathbb{F}_q$ is a finite field, we introduce a family $\{C_{\mathbb{X}}(d)\}_{d \in \mathbb{N}}$ of projective Reed-Muller-type codes that we call *projective Segre codes* (see Definition 2.5). It turns out that $C_{\mathbb{X}}(d)$ is isomorphic to $K[\mathbf{t}]_d/I(\mathbb{X})_d$, as K -vector spaces, where $I(\mathbb{X})_d$ is equal to $I(\mathbb{X}) \cap K[\mathbf{t}]_d$. Accordingly $C_{\mathbb{X}_1}(d) \simeq K[\mathbf{x}]_d/I(\mathbb{X}_1)_d$ and $C_{\mathbb{X}_2}(d) \simeq K[\mathbf{y}]_d/I(\mathbb{X}_2)_d$. In Section 3 we study the basic parameters (length, dimension, minimum distance) and the second generalized Hamming weight of projective Segre codes. Our main result expresses the basic parameters of $C_{\mathbb{X}}(d)$ in terms of those of $C_{\mathbb{X}_1}(d)$ and $C_{\mathbb{X}_2}(d)$, and shows that $C_{\mathbb{X}}(d)$ is the direct product of $C_{\mathbb{X}_1}(d)$ and $C_{\mathbb{X}_2}(d)$ (see Theorem 3.1); this means that the direct product of two projective Reed-Muller-type codes of degree d is again a projective Reed-Muller-type code of degree d .

Formulas for the basic parameters of affine and projective Reed-Muller-type codes are known for a number of families [4, 6, 7, 8, 11, 13, 14, 17, 18, 22, 28, 30]. Since affine Reed-Muller-type codes can be regarded as projective Reed-Muller-type codes [23], our results can be applied to obtain explicit formulas for the basic parameters of $C_{\mathbb{X}}(d)$ if $C_{\mathbb{X}_1}(d)$ is in one of these families and $C_{\mathbb{X}_2}(d)$ is in another of these families or both are in the same family.

As an application we recover some results on Reed-Muller-type codes over the Segre variety and over projective tori [14, 15, 16, 18]. Indeed, if $\mathbb{X}_1 = \mathbb{P}^{a_1-1}$ and $\mathbb{X}_2 = \mathbb{P}^{a_2-1}$, using Theorem 3.1 we recover the formula for the minimum distance of $C_{\mathbb{X}}(d)$ given in [18, Theorem 5.1]. If $K^* = K \setminus \{0\}$ and \mathbb{X}_i is the image of $(\mathbb{K}^*)^{a_i}$, under the map $(K^*)^{a_i} \rightarrow \mathbb{P}^{a_i-1}$, $x \rightarrow [x]$, we call \mathbb{X}_i a *projective torus* in \mathbb{P}^{a_i-1} . If \mathbb{X}_i is a projective torus for $i = 1, 2$, using Theorem 3.1 we recover the formula for the minimum distance of $C_{\mathbb{X}}(d)$ given in [14, Theorem 5.5]. In these two cases formulas for the basic parameters of $C_{\mathbb{X}_i}(d)$, $i = 1, 2$, are given in [30, Theorem 1] and [28, Theorem 3.5], respectively. We also recover the formulas for the second generalized Hamming weight given in [15, Theorem 5.1] and [16, Theorem 3] (see Corollary 3.5).

For all unexplained terminology and notation, and for additional information we refer to [5, 31] (for the theory of Hilbert functions) and to [25, 32] (for coding theory). Our main references for commutative algebra and multilinear algebra are [2, 19] and [9, Appendix 2], respectively.

2. PRELIMINARIES

In this section, we present some of the results that will be needed throughout the paper and introduce some more notation. We study direct product codes, and some of their properties and characterizations. The families of Reed-Muller-type codes and projective Segre codes are introduced here, and their relation to tensor products and Hilbert functions is discussed.

Generalized Hamming weights. Let $K = \mathbb{F}_q$ be a finite field and let C be a $[s, k]$ linear code of length s and dimension k , that is, C is a linear subspace of K^s with $k = \dim_K(C)$.

Given a subcode D of C (that is, D is a linear subspace of C), the *support* of D , denoted $\chi(D)$, is the set of non-zero positions of D , that is,

$$\chi(D) := \{i \mid \exists (a_1, \dots, a_s) \in D, a_i \neq 0\}.$$

The r th *generalized Hamming weight* of C , denoted $\delta_r(C)$, is the size of the smallest support of an r -dimensional subcode, that is,

$$\delta_r(C) := \min\{|\chi(D)| : D \text{ is a linear subcode of } C \text{ with } \dim_K(D) = r\}.$$

Let $0 \neq v \in C$. The *Hamming weight* of v , denoted by $\omega(v)$, is the number of non-zero entries of v . If $\delta(C)$ is the *minimum distance* of C , that is, $\delta(C) := \min\{\omega(v) : 0 \neq v \in C\}$, then note that $\delta_1(C) = \delta(C)$. The *weight hierarchy* of C is the sequence $(\delta_1(C), \dots, \delta_k(C))$. According to [34, Theorem 1] the weight hierarchy is an increasing sequence

$$1 \leq \delta_1(C) < \delta_2(C) < \dots < \delta_r(C) \leq s,$$

and $\delta_r(C) \leq s - k + r$ for $r = 1, \dots, k$. For $r = 1$ this is the Singleton bound for the minimum distance. Generalized Hamming weights have received a lot of attention; see [3, 10, 29, 34, 35] and the references therein.

Direct product codes and tensor products. Let $C_1 \subset K^{s_1}$ and $C_2 \subset K^{s_2}$ be two linear codes over the finite field $K = \mathbb{F}_q$ and let $M_{s_1 \times s_2}(K)$ be the K -vector space of all matrices of size $s_1 \times s_2$ with entries in K .

The *direct product* (also called *Kronecker product*) of C_1 and C_2 , denoted by $C_1 \underline{\otimes} C_2$, is defined to be the linear code consisting of all $s_1 \times s_2$ matrices in which the rows belong to C_2 and the columns to C_1 ; see [32, p. 44]. The direct product codes usually have poor minimum distance but are easy to decode and can be useful in certain applications; see [25, Chapter 18].

We denote the tensor product of C_1 and C_2 —in the sense of multilinear algebra [9, p. 573]—by $C_1 \otimes_K C_2$. As is shown in Lemma 2.3 another way to see the direct product of C_1 and C_2 is as a tensor product.

Theorem 2.1. [33, Theorems 2.5.2 and 2.5.3] *Let $C_i \subset K^{s_i}$ be a linear code of length s_i , dimension k_i , and minimum distance $\delta(C_i)$ for $i = 1, 2$. Then $C_1 \underline{\otimes} C_2$ has length $s_1 s_2$, dimension $k_1 k_2$, and minimum distance $\delta(C_1) \delta(C_2)$.*

Theorem 2.2. [35, Theorem 3(d)] *Let $C_1 \subset K^{s_1}$ and $C_2 \subset K^{s_2}$ be two linear codes and let $C = C_1 \underline{\otimes} C_2$ be their direct product. Then*

$$\delta_2(C) = \min\{\delta_1(C_1)\delta_2(C_2), \delta_2(C_1)\delta_1(C_2)\}.$$

Recall that there is a natural isomorphism $\text{vec}: M_{s_1 \times s_2}(K) \rightarrow K^{s_1 s_2}$ of K -vector spaces given by $\text{vec}(A) = (F_1, \dots, F_{s_1})$, where F_1, \dots, F_{s_1} are the rows of A . Consider the bilinear map ψ_0 given by

$$\begin{aligned} \psi_0: K^{s_1} \times K^{s_2} &\longrightarrow M_{s_1 \times s_2}(K) \\ ((a_1, \dots, a_{s_1}), (b_1, \dots, b_{s_2})) &\longmapsto \begin{bmatrix} a_1 b_1 & a_1 b_2 & \dots & a_1 b_{s_2} \\ a_2 b_1 & a_2 b_2 & \dots & a_2 b_{s_2} \\ \vdots & \vdots & & \vdots \\ a_{s_1} b_1 & a_{s_1} b_2 & \dots & a_{s_1} b_{s_2} \end{bmatrix}. \end{aligned}$$

The Segre embedding, defined in the introduction, is given by $\psi([a], [b]) = [(\text{vec} \circ \psi_0)(a, b)]$, where $a = (a_1, \dots, a_{s_1})$ and $b = (b_1, \dots, b_{s_2})$.

The next lemma is not hard to prove and probably known in some equivalent formulation; but we could not find a reference with the corresponding proof.

Lemma 2.3. *There is an isomorphism $T: C_1 \otimes_K C_2 \rightarrow C_1 \underline{\otimes} C_2$ of K -vector spaces such that $T(a \otimes b) = \psi_0(a, b)$ for $a \in C_1$ and $b \in C_2$.*

Proof. We set $k_i = \dim_K(C_i)$ for $i = 1, 2$. Using the universal property of the tensor product [9, p. 573], we get that the bilinear map ψ_0 induces a linear map

$$\begin{aligned} T: C_1 \otimes_K C_2 &\longrightarrow C_1 \underline{\otimes} C_2, \text{ such that,} \\ a \otimes b &\longmapsto \psi_0(a, b) \end{aligned}$$

for $a \in C_1$ and $b \in C_2$. By [26, Formula 5, p. 267] and Theorem 2.1, one has that $C_1 \otimes_K C_2$ and $C_1 \underline{\otimes} C_2$ have dimension $k_1 k_2$. Thus to prove that T is an isomorphism it suffices to prove that T is a one-to-one linear map. Fix bases $\{\alpha_1, \dots, \alpha_{k_1}\}$ and $\{\beta_1, \dots, \beta_{k_2}\}$ of C_1 and C_2 , respectively. Take any element γ in the kernel of T . We can write

$$\gamma = \sum \lambda_{i,j} \alpha_i \otimes \beta_j$$

with $\lambda_{i,j}$ in K for all i, j . Then

$$\begin{aligned} T(\gamma) &= \lambda_{1,1} T(\alpha_1 \otimes \beta_1) + \dots + \lambda_{1,k_2} T(\alpha_1 \otimes \beta_{k_2}) + \\ &\quad \lambda_{2,1} T(\alpha_2 \otimes \beta_1) + \dots + \lambda_{2,k_2} T(\alpha_2 \otimes \beta_{k_2}) + \\ &\quad \vdots \\ &\quad \lambda_{k_1,1} T(\alpha_{k_1} \otimes \beta_1) + \dots + \lambda_{k_1,k_2} T(\alpha_{k_1} \otimes \beta_{k_2}). \end{aligned}$$

Setting $\alpha_i = (\alpha_{i,1}, \dots, \alpha_{i,s_1})$, $\beta_j = (\beta_{j,1}, \dots, \beta_{j,s_2})$ for $i = 1, \dots, k_1$, $j = 1, \dots, k_2$, we get

$$T(\gamma) = \begin{bmatrix} (\lambda_{1,1}\alpha_{1,1}\beta_1 + \dots + \lambda_{1,k_2}\alpha_{1,1}\beta_{k_2}) + \dots + (\lambda_{k_1,1}\alpha_{k_1,1}\beta_1 + \dots + \lambda_{k_1,k_2}\alpha_{k_1,1}\beta_{k_2}) \\ (\lambda_{1,1}\alpha_{1,2}\beta_1 + \dots + \lambda_{1,k_2}\alpha_{1,2}\beta_{k_2}) + \dots + (\lambda_{k_1,1}\alpha_{k_1,2}\beta_1 + \dots + \lambda_{k_1,k_2}\alpha_{k_1,2}\beta_{k_2}) \\ \vdots \\ (\lambda_{1,1}\alpha_{1,s_1}\beta_1 + \dots + \lambda_{1,k_2}\alpha_{1,s_1}\beta_{k_2}) + \dots + (\lambda_{k_1,1}\alpha_{k_1,s_1}\beta_1 + \dots + \lambda_{k_1,k_2}\alpha_{k_1,s_1}\beta_{k_2}) \end{bmatrix}.$$

Since $T(\gamma) = (0)$, using that the β_i 's are linearly independent, we get

$$\lambda_{1,j}\alpha_1^\top + \dots + \lambda_{k_1,j}\alpha_{k_1}^\top = 0 \text{ for } j = 1, \dots, k_2.$$

Thus $\lambda_{i,j} = 0$ for all i, j and $\gamma = 0$. □

Hilbert functions. Let K be a field. Recall that the *projective space* of dimension $s - 1$ over K , denoted by \mathbb{P}^{s-1} , is the quotient space

$$(K^s \setminus \{0\}) / \sim$$

where two points α, β in $K^s \setminus \{0\}$ are equivalent under \sim if $\alpha = \lambda\beta$ for some $\lambda \in K^*$. We denote the equivalence class of α by $[\alpha]$.

Let $X \neq \emptyset$ be a subset of \mathbb{P}^{s-1} . Consider a graded polynomial ring $S = K[t_1, \dots, t_s]$, over the field K , where each t_i is homogeneous of degree one. Let S_d denote the set of homogeneous polynomials of total degree d in S , together with the zero polynomial, and let $I(X)$ be the *vanishing ideal* of X generated by the homogeneous polynomials of S that vanish at all points of X . The set S_d is a K -vector space of dimension $\binom{d+s-1}{s-1}$. We let

$$I(X)_d := I(X) \cap S_d,$$

denote the set of homogeneous polynomials in $I(X)$ of total degree d , together with the zero polynomial. Note that $I(X)_d$ is a vector subspace of S_d . The *Hilbert function* of the quotient ring $S/I(X)$, denoted by $H_X(d)$, is defined as

$$H_X(d) := \dim_K(S_d/I(X)_d).$$

According to a classical result of Hilbert [2, Theorem 4.1.3], there is a unique polynomial

$$h_X(t) = c_k t^k + (\text{terms of lower degree})$$

of degree $k \geq 0$, with rational coefficients, such that $h_X(d) = H_X(d)$ for $d \gg 0$. The integer $k+1$ is the *Krull dimension* of $S/I(X)$, k is the *dimension* of X , and $h_X(t)$ is the *Hilbert polynomial* of $S/I(X)$. The positive integer $c_k(k!)$ is the *degree* of $S/I(X)$. The *index of regularity* of $S/I(X)$, denoted by $\text{reg}(S/I(X))$, is the least integer $r \geq 0$ such that $h_X(d) = H_X(d)$ for $d \geq r$. The degree and the Krull dimension are denoted by $\deg(S/I(X))$ and $\dim(S/I(X))$, respectively.

Proposition 2.4. ([8], [12], [24]) *If X is a finite set and $r = \text{reg}(S/I(X))$, then*

$$1 = H_X(0) < H_X(1) < \cdots < H_X(r-1) < H_X(d) = \deg(S/I(X)) = |X| \quad \text{for } d \geq r.$$

Projective Reed-Muller-type codes. In this part we introduce the family of projective Reed-Muller-type codes and its connection to vanishing ideals and Hilbert functions.

Let $K = \mathbb{F}_q$ be a finite field and let $X = \{P_1, \dots, P_m\} \neq \emptyset$ be a subset of \mathbb{P}^{s-1} with $m = |X|$. Fix a degree $d \geq 0$. For each i there is $f_i \in S_d$ such that $f_i(P_i) \neq 0$; we refer to Section 3 to see a convenient way to choose f_1, \dots, f_m . There is a well-defined K -linear map given by

$$(2.1) \quad \text{ev}_d: S_d = K[t_1, \dots, t_s]_d \rightarrow K^{|X|}, \quad f \mapsto \left(\frac{f(P_1)}{f_1(P_1)}, \dots, \frac{f(P_m)}{f_m(P_m)} \right).$$

The map ev_d is called an *evaluation map*. The image of S_d under ev_d , denoted by $C_X(d)$, is called a *projective Reed-Muller-type code* of degree d over the set X [8, 18]. It is also called an *evaluation code* associated to X [13]. The kernel of the evaluation map ev_d is $I(X)_d$. Hence there is an isomorphism of K -vector spaces $S_d/I(X)_d \simeq C_X(d)$.

Definition 2.5. If \mathbb{X} is the Segre product of \mathbb{X}_1 and \mathbb{X}_2 , we say that $C_{\mathbb{X}}(d)$ is a *projective Segre code* of degree d ; recall that \mathbb{X} is the image of $\mathbb{X}_1 \times \mathbb{X}_2$ under the Segre embedding ψ .

Definition 2.6. The *basic parameters* of the linear code $C_X(d)$ are:

- (a) *length*: $|X|$,
- (b) *dimension*: $\dim_K C_X(d)$, and
- (c) *minimum distance*: $\delta(C_X(d))$. We also denote $\delta(C_X(d))$ simply by $\delta_X(d)$.

The basic parameters of projective Reed-Muller-type codes have been computed in a number of cases. If $X = \mathbb{P}^{s-1}$ then $C_X(d)$ is the *classical projective Reed-Muller code* and its basic parameters are described in [30, Theorem 1]. If X is a projective torus, $C_X(d)$ is the *generalized projective Reed-Solomon code* and its basic parameters are described in [28, Theorem 3.5].

The following summarizes the well-known relation between projective Reed-Muller-type codes and the theory of Hilbert functions.

Proposition 2.7. ([18], [27]) *The following hold.*

- (i) $H_X(d) = \dim_K C_X(d)$ for $d \geq 0$.
- (ii) $\delta_X(d) = 1$ for $d \geq \text{reg}(S/I(X))$.
- (iii) $S/I(X)$ is a Cohen-Macaulay graded ring of dimension 1.
- (iv) $C_X(d) \neq (0)$ for $d \geq 0$.

Segre products. To avoid repetitions, we continue to employ the notations and definitions used in Section 1. For the rest of this section K will denote an arbitrary field.

A *standard algebra* over a field K is a finitely generated graded K -algebra $A = \bigoplus_{d=0}^{\infty} A_d$ such that $A = K[A_1]$ and $A_0 = K$ (that is, A is isomorphic to $K[\mathbf{x}]/I$, for some polynomial ring $K[\mathbf{x}]$ with the standard grading and for some graded ideal I).

Definition 2.8. [9, p. 304] Let $A = \bigoplus_{d \geq 0} A_d$, $B = \bigoplus_{d \geq 0} B_d$ be two standard algebras over a field K . The *Segre product* of A and B , denoted by $A \otimes_S B$, is the graded algebra

$$A \otimes_S B := (A_0 \otimes_K B_0) \oplus (A_1 \otimes_K B_1) \oplus \cdots \subset A \otimes_K B,$$

with the normalized grading $(A \otimes_S B)_d := A_d \otimes_K B_d$ for $d \geq 0$. The tensor product algebra $A \otimes_K B$ is graded by

$$(A \otimes_K B)_p := \sum_{i+j=p} A_i \otimes_K B_j.$$

Example 2.9. [1, p. 161] The Segre product (resp. tensor product) of $K[\mathbf{x}]$ and $K[\mathbf{y}]$ is

$$K[\mathbf{x}] \otimes_S K[\mathbf{y}] \simeq K[\{x_i y_j \mid 1 \leq i \leq a_1, 1 \leq j \leq a_2\}]$$

(resp. $K[\mathbf{x}] \otimes_K K[\mathbf{y}] \simeq K[\mathbf{x}, \mathbf{y}]$). Notice that the elements $x_i y_j$ have normalized degree 1 as elements of $K[\mathbf{x}] \otimes_S K[\mathbf{y}]$ and total degree 2 as elements of $K[\mathbf{x}] \otimes_K K[\mathbf{y}]$.

The next result is well-known assuming that \mathbb{X}_1 and \mathbb{X}_2 are projective algebraic sets; see for instance [9, Exercise 13.14(d)]. However David Eisenbud pointed out to us that the result is valid in general. We give a proof of the general case.

Theorem 2.10. *Let K be a field. If $\mathbb{X}_1, \mathbb{X}_2$ are subsets of the projective spaces $\mathbb{P}^{a_1-1}, \mathbb{P}^{a_2-1}$, respectively, and \mathbb{X} is the Segre product of \mathbb{X}_1 and \mathbb{X}_2 , then the following hold:*

- (a) $(K[\mathbf{x}]/I(\mathbb{X}_1))_d \otimes_K (K[\mathbf{y}]/I(\mathbb{X}_2))_d \simeq (K[\mathbf{t}]/I(\mathbb{X}))_d$ as K -vector spaces for $d \geq 0$.
- (b) $K[\mathbf{x}]/I(\mathbb{X}_1) \otimes_S K[\mathbf{y}]/I(\mathbb{X}_2) \simeq K[\mathbf{t}]/I(\mathbb{X})$ as standard graded algebras.
- (c) $H_{\mathbb{X}_1}(d) H_{\mathbb{X}_2}(d) = H_{\mathbb{X}}(d)$ for $d \geq 0$.
- (d) $\text{reg}(K[\mathbf{t}]/I(\mathbb{X})) = \max\{\text{reg}(K[\mathbf{x}]/I(\mathbb{X}_1)), \text{reg}(K[\mathbf{y}]/I(\mathbb{X}_2))\}$.
- (e) If $\rho_1 = \dim(K[\mathbf{x}]/I(\mathbb{X}_1))$ and $\rho_2 = \dim(K[\mathbf{y}]/I(\mathbb{X}_2))$, then

$$\deg(K[\mathbf{t}]/I(\mathbb{X})) = \deg(K[\mathbf{x}]/I(\mathbb{X}_1)) \deg(K[\mathbf{y}]/I(\mathbb{X}_2)) \binom{\rho_1 + \rho_2 - 2}{\rho_1 - 1}.$$

Proof. (a): Let σ be the epimorphism of K -algebras $\sigma: K[\mathbf{t}] \rightarrow K[\{x_i y_j \mid i \in [1, a_1], j \in [1, a_2]\}]$ induced by $t_{ij} \mapsto x_i y_j$, where $[1, a_i] = \{1, \dots, a_i\}$. For each $x^b y^c$ with $\deg(x^b) = \deg(y^c) = d$ there is a unique $t^a \in K[\mathbf{t}]_d$ such that $t^a = t_{i_1, j_1} \cdots t_{i_d, j_d}$, $1 \leq i_1 \leq \dots \leq i_d$, $1 \leq j_1 \leq \dots \leq j_d$ and $\sigma(t^a) = x^b y^c$. Notice that if $\sigma(t^\alpha) = x^b y^c$ for some other monomial $t^\alpha \in K[\mathbf{t}]_d$, then $t^a - t^\alpha \in I(\mathbb{X})$. This is used below to ensure that the mapping of Eq. (2.2) is surjective. Setting $\varphi_0(x^b, y^c) = t^a$, we get a K -bilinear map

$$\varphi_0: K[\mathbf{x}]_d \times K[\mathbf{y}]_d \rightarrow K[\mathbf{t}]_d$$

induced by $\varphi_0(x^b, y^c) = t^a$. Notice that $\varphi_0(\sum \lambda_k x^{b_k}, \sum \mu_\ell y^{c_\ell}) = \sum \lambda_k \mu_\ell \varphi_0(x^{b_k}, y^{c_\ell})$, where the λ_k 's and μ_ℓ 's are in K . To show that φ_0 induces a K -bilinear map

$$(2.2) \quad \varphi: (K[\mathbf{x}]_d/I(\mathbb{X}_1)_d) \times (K[\mathbf{y}]_d/I(\mathbb{X}_2)_d) \rightarrow K[\mathbf{t}]_d/I(\mathbb{X})_d, \quad (\overline{x^b}, \overline{y^c}) \mapsto \overline{\varphi_0(x^b, y^c)},$$

which is a surjection, it suffices to show that for any $f \in K[\mathbf{x}]_d$ that vanish on \mathbb{X}_1 (resp. $g \in K[\mathbf{y}]_d$ that vanish on \mathbb{X}_2) one has that $\varphi_0(f, g)$ vanishes at all points of \mathbb{X} . Assume that $f = \lambda_1 x^{b_1} + \dots + \lambda_m x^{b_m}$ is a polynomial in $K[\mathbf{x}]_d$ that vanish on \mathbb{X}_1 and that $g = \mu_1 y^{c_1} + \dots + \mu_r y^{c_r}$

is a polynomial in $K[\mathbf{y}]_d$ with λ_k, μ_ℓ in K for all k, ℓ . For each $x^{b_k}y^{c_\ell}$ there is $t^{a_{k\ell}} \in K[\mathbf{t}]$ such that $\sigma(t^{a_{k\ell}}) = x^{b_k}y^{c_\ell}$. Then

$$\begin{aligned}\varphi_0(f, g) &= \sum \lambda_k \mu_\ell \varphi_0(x^{b_k}, y^{c_\ell}) = \sum \lambda_k \mu_\ell t^{a_{k\ell}}, \text{ and} \\ \varphi_0(f, g)(x_i y_j) &= (\lambda_1 x^{b_1} + \cdots + \lambda_m x^{b_m})(\mu_1 y^{c_1} + \cdots + \mu_r y^{c_r}),\end{aligned}$$

where we use $(x_i y_j)$ as a short hand for $(x_1 y_1, x_1 y_2, \dots, x_1 y_{a_2}, \dots, x_{a_1} y_1, x_{a_1} y_2, \dots, x_{a_1} y_{a_2})$. Now if $[(\alpha_1, \dots, \alpha_{a_1})]$ is in \mathbb{X}_1 and $[(\beta_1, \dots, \beta_{a_2})]$ is in \mathbb{X}_2 , making $x_i = \alpha_i$ and $y_j = \beta_j$ for all i, j in the last equality, we get $\varphi_0(f, g)(\alpha_i \beta_j) = 0$. Therefore, by the universal property of the tensor product [9, p. 573], there is a surjective map $\bar{\varphi}$ that makes the following diagram commutative:

$$\begin{array}{ccc}(K[\mathbf{x}]_d/I(\mathbb{X}_1)_d) \times (K[\mathbf{y}]_d/I(\mathbb{X}_2)_d) & \xrightarrow{\phi} & (K[\mathbf{x}]_d/I(\mathbb{X}_1)_d) \otimes_K (K[\mathbf{y}]_d/I(\mathbb{X}_2)_d) \\ \downarrow \varphi & & \swarrow \bar{\varphi} \\ K[\mathbf{t}]_d/I(\mathbb{X})_d & & \end{array}$$

where ϕ is the canonical map, given by $\phi(\bar{f}, \bar{g}) = \bar{f} \otimes \bar{g}$, and $\varphi = \bar{\varphi}\phi$.

For each $t^\alpha \in K[\mathbf{t}]_d$ let $x^b \in K[\mathbf{x}]_d$ and $y^c \in K[\mathbf{y}]_d$ be such that $\sigma(t^\alpha) = x^b y^c$. We set $\sigma_1(t^\alpha) = x^b$ and $\sigma_2(t^\alpha) = y^c$. Thus we have a surjective K -linear map

$$\sigma_0^*: K[\mathbf{t}]_d \rightarrow K[\mathbf{x}]_d/I(\mathbb{X}_1)_d \otimes_K K[\mathbf{y}]_d/I(\mathbb{X}_2)_d$$

given by $\sigma_0^*(\sum \lambda_\alpha t^\alpha) = \sum \lambda_\alpha \overline{\sigma_1(t^\alpha)} \otimes \overline{\sigma_2(t^\alpha)}$, where the λ_α 's are in K . Notice that the K -vector space on the right hand side is generated by all $\bar{x}^b \otimes \bar{y}^c$ such that $x^b \in K[\mathbf{x}]_d$ and $y^c \in K[\mathbf{y}]_d$. Take $f \in I(\mathbb{X})_d$, then $\sigma(f)(\alpha_i \beta_j) = 0$ for all $\alpha = [(\alpha_1, \dots, \alpha_{a_1})] \in \mathbb{X}_1$ and all $\beta = [(\beta_1, \dots, \beta_{a_2})] \in \mathbb{X}_2$. We can write $\sigma(f) = \sum_{\ell=1}^k f_\ell g_\ell$ with $f_\ell \in K[\mathbf{x}]_d$, $g_\ell \in K[\mathbf{y}]_d$ for $\ell = 1, \dots, k$, and $\sigma_0^*(f) = \sum_{\ell=1}^k \bar{f}_\ell \otimes \bar{g}_\ell$. Next we show, by induction on k , that $\sigma_0^*(f) = 0$, i.e., $f \in \ker(\sigma_0^*)$. If $k = 1$, we may assume that $f_1 \notin I(\mathbb{X}_1)$ otherwise $\bar{f}_1 = \bar{0}$. Pick $\alpha \in \mathbb{X}_1$ such that $f_1(\alpha) \neq 0$. Then, as $f_1(\alpha)g_1(\beta) = 0$ for all $\beta \in \mathbb{X}_2$, one has $g_1 \in I(\mathbb{X}_2)$ and $\bar{g}_1 = \bar{0}$. We may now assume that $k > 1$ and $\bar{f}_k \neq 0$. Pick $\alpha \in \mathbb{X}_1$ such that $f_k(\alpha) \neq 0$. By hypothesis the polynomial

$$f_1(\alpha)g_1 + \cdots + f_k(\alpha)g_k$$

is in $K[\mathbf{y}]_d$ and vanishes at all points of \mathbb{X}_2 . Thus

$$\bar{g}_k = -(f_1(\alpha)/f_k(\alpha))\bar{g}_1 - \cdots - (f_{k-1}(\alpha)/f_k(\alpha))\bar{g}_{k-1}.$$

Therefore, setting $h_\ell = f_\ell - (f_\ell(\alpha)/f_k(\alpha))f_k$ for $\ell = 1, \dots, k-1$, we get

$$\sigma_0^*(f) = \sum_{\ell=1}^k \bar{f}_\ell \otimes \bar{g}_\ell = \sum_{\ell=1}^{k-1} \bar{h}_\ell \otimes \bar{g}_\ell$$

and $\sum_{\ell=1}^{k-1} h_\ell(\gamma)g_\ell(\beta) = 0$ for all $\gamma \in \mathbb{X}_1$ and $\beta \in \mathbb{X}_2$. Thus, by induction, $\sigma_0^*(f) = 0$. Hence $I(\mathbb{X})_d \subset \ker(\sigma_0^*)$. Therefore σ_0^* induces a K -linear surjection

$$\sigma^*: K[\mathbf{t}]_d/I(\mathbb{X})_d \rightarrow (K[\mathbf{x}]_d/I(\mathbb{X}_1)_d) \otimes_K (K[\mathbf{y}]_d/I(\mathbb{X}_2)_d).$$

Altogether we get that the linear maps $\bar{\varphi}$ and σ^* are bijective.

Items (b) to (e) follow directly from (a) and its proof. \square

3. BASIC PARAMETERS OF PROJECTIVE SEGRE CODES

In this section we study projective Segre codes and their basic parameters; including the second generalized Hamming weight. It is shown that direct product codes of projective Reed-Muller-type codes are projective Segre codes. Then some applications are given. We continue to employ the notations and definitions used in Sections 1 and 2.

In preparation for our main theorem, let $K = \mathbb{F}_q$ be a finite field, let a_1, a_2 be two positive integers with $a_1 \geq a_2$, and for $i = 1, 2$, let \mathbb{X}_i be a non-empty subset of the projective space \mathbb{P}^{a_i-1} over K . We set $s = a_1 a_2$ and $s_i = |\mathbb{X}_i|$ for $i = 1, 2$. The *Segre embedding* is given by

$$\begin{aligned} \psi: \mathbb{P}^{a_1-1} \times \mathbb{P}^{a_2-1} &\rightarrow \mathbb{P}^{a_1 a_2 - 1} = \mathbb{P}^{s-1} \\ ([(\alpha_1, \dots, \alpha_{a_1})], [(\beta_1, \dots, \beta_{a_2})]) &\rightarrow [(\alpha_1 \beta_1, \alpha_1 \beta_2, \dots, \alpha_1 \beta_{a_2}, \\ &\quad \alpha_2 \beta_1, \alpha_2 \beta_2, \dots, \alpha_2 \beta_{a_2}, \\ &\quad \vdots \\ &\quad \alpha_{a_1} \beta_1, \alpha_{a_1} \beta_2, \dots, \alpha_{a_1} \beta_{a_2})]. \end{aligned}$$

The image of $\mathbb{X}_1 \times \mathbb{X}_2$ under the map ψ , denoted by \mathbb{X} , is the *Segre product* of \mathbb{X}_1 and \mathbb{X}_2 . As ψ is injective, we get $|\mathbb{X}| = |\mathbb{X}_1| |\mathbb{X}_2| = s_1 s_2$. Then we can write \mathbb{X} , \mathbb{X}_1 , and \mathbb{X}_2 as:

$$\begin{aligned} \mathbb{X} = \{P_{1,1}, \dots, P_{s_1, s_2}\} &= \{P_{1,1}, P_{1,2}, \dots, P_{1, s_2}, \\ &\quad P_{2,1}, P_{2,2}, \dots, P_{2, s_2}, \\ &\quad \vdots \\ &\quad P_{s_1,1}, P_{s_1,2}, \dots, P_{s_1, s_2}\}, \end{aligned}$$

$\mathbb{X}_1 = \{Q_1, \dots, Q_{s_1}\}$, and $\mathbb{X}_2 = \{R_1, \dots, R_{s_2}\}$, respectively, where

$$Q_i = [(\alpha_{i,1}, \alpha_{i,2}, \dots, \alpha_{i,a_1})] \quad \text{and} \quad R_j = [(\beta_{j,1}, \beta_{j,2}, \dots, \beta_{j,a_2})],$$

for $i = 1, \dots, s_1$ and $j = 1, \dots, s_2$. Because of the embedding ψ each $P_{i,j} \in \mathbb{X}$ is of the form

$$\begin{aligned} P_{i,j} = \psi(Q_i, R_j) &= [(\alpha_{i,1} \cdot \beta_{j,1}, \alpha_{i,1} \cdot \beta_{j,2}, \dots, \alpha_{i,1} \cdot \beta_{j,a_2}, \\ &\quad \alpha_{i,2} \cdot \beta_{j,1}, \alpha_{i,2} \cdot \beta_{j,2}, \dots, \alpha_{i,2} \cdot \beta_{j,a_2}, \\ &\quad \vdots \\ &\quad \alpha_{i,a_1} \cdot \beta_{j,1}, \alpha_{i,a_1} \cdot \beta_{j,2}, \dots, \alpha_{i,a_1} \cdot \beta_{j,a_2})]. \end{aligned}$$

For use below notice that for each $i \in \llbracket 1, s_1 \rrbracket$ and for each $j \in \llbracket 1, s_2 \rrbracket$ there are $k_i \in \llbracket 1, a_1 \rrbracket$ and $\ell_j \in \llbracket 1, a_2 \rrbracket$ such that $\alpha_{i,k_i} \neq 0$ and $\beta_{j,\ell_j} \neq 0$. In fact, choose k_i to be the smallest $k \in \llbracket 1, a_1 \rrbracket$ such that $\alpha_{i,k} \neq 0$, and choose ℓ_j to be the smallest $\ell \in \llbracket 1, a_2 \rrbracket$ such that $\beta_{j,\ell} \neq 0$. Hence $\alpha_{i,k_i} \cdot \beta_{j,\ell_j} \neq 0$.

Setting $K[\mathbf{t}] = K[t_{1,1}, t_{1,2}, \dots, t_{1,a_2}, \dots, t_{a_1,1}, t_{a_1,2}, \dots, t_{a_1,a_2}]$ and fixing an integer $d \geq 1$, define $f_{i,j}(t_{1,1}, \dots, t_{a_1,a_2}) = (t_{k_i, \ell_j})^d$. Then $f_{i,j}(P_{i,j}) = (\alpha_{i,k_i} \cdot \beta_{j,\ell_j})^d \neq 0$. The evaluation map ev_d is defined as:

$$\begin{aligned} \text{ev}_d: K[\mathbf{t}]_d &\rightarrow K^{|\mathbb{X}|} = K^{s_1 s_2}, \\ f &\rightarrow \left(\frac{f(P_{1,1})}{f_{1,1}(P_{1,1})}, \frac{f(P_{1,2})}{f_{1,2}(P_{1,2})}, \dots, \frac{f(P_{s_1, s_2})}{f_{s_1, s_2}(P_{s_1, s_2})} \right). \end{aligned}$$

This is a linear map of K -vector spaces. The image of ev_d , denoted by $C_{\mathbb{X}}(d)$, defines a projective Reed-Muller-type linear code of degree d that we call a *projective Segre code* of degree d .

For each $i \in \llbracket 1, s_1 \rrbracket$ and for each $j \in \llbracket 1, s_2 \rrbracket$, define the following polynomials:

$$g_i(x_1, \dots, x_{a_1}) = x_{k_i}^d \in K[x_1, \dots, x_{a_1}]_d \quad \text{and} \quad h_j(y_1, \dots, y_{a_2}) = y_{\ell_j}^d \in K[y_1, \dots, y_{a_2}]_d.$$

Clearly $g_i(Q_i) = \alpha_{i,k_i}^d \neq 0$, $h_j(R_j) = \beta_{j,\ell_j}^d \neq 0$, $f_{i,j}(P_{i,j}) = (\alpha_{i,k_i})^d h_j(R_j) = g_i(Q_i)(\beta_{j,\ell_j})^d$. We also define the following two evaluation maps:

$$\begin{aligned} \text{ev}_d^1: K[x_1, \dots, x_{a_1}]_d &\rightarrow K^{|\mathbb{X}_1|} = K^{s_1}, \\ g &\rightarrow \left(\frac{g(Q_1)}{g_1(Q_1)}, \frac{g(Q_2)}{g_2(Q_2)}, \dots, \frac{g(Q_{s_1})}{g_{s_1}(Q_{s_1})} \right), \text{ and} \\ \text{ev}_d^2: K[y_1, \dots, y_{a_2}]_d &\rightarrow K^{|\mathbb{X}_2|} = K^{s_2}, \\ h &\rightarrow \left(\frac{h(R_1)}{h_1(R_1)}, \frac{h(R_2)}{h_2(R_2)}, \dots, \frac{h(R_{s_2})}{h_{s_2}(R_{s_2})} \right), \end{aligned}$$

and their corresponding Reed-Muller-type linear codes $C_{\mathbb{X}_i}(d) := \text{im}(\text{ev}_d^i)$ for $i = 1, 2$.

Let C be a linear code. From Section 2 recall that $\delta_r(C)$ is the r th generalized Hamming weight of C and that $\delta_1(C)$ is the minimum distance $\delta(C)$ of C .

We come to the main result of this section.

Theorem 3.1. *Let $K = \mathbb{F}_q$ be a finite field, let $\mathbb{X}_i \subset \mathbb{P}^{a_i-1}$ for $i = 1, 2$, and let \mathbb{X} be the Segre product of \mathbb{X}_1 and \mathbb{X}_2 . The following hold.*

- (a) $|\mathbb{X}| = |\mathbb{X}_1||\mathbb{X}_2|$.
- (b) $\dim_K(C_{\mathbb{X}}(d)) = \dim_K(C_{\mathbb{X}_1}(d)) \dim_K(C_{\mathbb{X}_2}(d))$ for $d \geq 1$.
- (c) $C_{\mathbb{X}}(d)$ is the direct product $C_{\mathbb{X}_1}(d) \otimes C_{\mathbb{X}_2}(d)$ of $C_{\mathbb{X}_1}(d)$ and $C_{\mathbb{X}_2}(d)$ for $d \geq 1$.
- (d) $\delta(C_{\mathbb{X}}(d)) = \delta(C_{\mathbb{X}_1}(d))\delta(C_{\mathbb{X}_2}(d))$ for $d \geq 1$.
- (e) $\delta_2(C_{\mathbb{X}}(d)) = \min\{\delta_1(C_{\mathbb{X}_1}(d))\delta_2(C_{\mathbb{X}_2}(d)), \delta_2(C_{\mathbb{X}_1}(d))\delta_1(C_{\mathbb{X}_2}(d))\}$ for $d \geq 1$.
- (f) $\delta(C_{\mathbb{X}}(d)) = 1$ for $d \geq \max\{\text{reg}(K[\mathbf{x}]/I(\mathbb{X}_1)), \text{reg}(K[\mathbf{y}]/I(\mathbb{X}_2))\}$.

Proof. (a): This is clear because the Segre embedding is a one-to-one map.

(b): Since $K[\mathbf{x}]_d/I(\mathbb{X}_1)_d \simeq C_{\mathbb{X}_1}(d)$, $K[\mathbf{y}]_d/I(\mathbb{X}_2)_d \simeq C_{\mathbb{X}_2}(d)$, and $K[\mathbf{t}]_d/I(\mathbb{X})_d \simeq C_{\mathbb{X}}(d)$, the results follows at once from Theorem 2.10.

(c): Given $f \in K[\mathbf{t}]_d$, the entries of $\text{ev}_d(f)$ can be arranged as:

$$(3.1) \quad \begin{aligned} \text{ev}_d(f) &= \begin{pmatrix} \frac{f(P_{1,1})}{f_{1,1}(P_{1,1})}, & \frac{f(P_{1,2})}{f_{1,2}(P_{1,2})}, & \dots, & \frac{f(P_{1,s_2})}{f_{1,s_2}(P_{1,s_2})}, & \rightarrow \Gamma_1 \\ \frac{f(P_{2,1})}{f_{2,1}(P_{2,1})}, & \frac{f(P_{2,2})}{f_{2,2}(P_{2,2})}, & \dots, & \frac{f(P_{2,s_2})}{f_{2,s_2}(P_{2,s_2})}, & \rightarrow \Gamma_2 \\ \vdots & \vdots & & \vdots & \vdots \\ \frac{f(P_{s_1,1})}{f_{s_1,1}(P_{s_1,1})}, & \frac{f(P_{s_1,2})}{f_{s_1,2}(P_{s_1,2})}, & \dots, & \frac{f(P_{s_1,s_2})}{f_{s_1,s_2}(P_{s_1,s_2})}, & \rightarrow \Gamma_{s_1} \end{pmatrix} \\ &\quad \begin{matrix} \downarrow & \downarrow & \dots & \downarrow \\ \Lambda_1 & \Lambda_2 & \dots & \Lambda_{s_2} \end{matrix} \end{aligned}$$

where $\Gamma_1, \dots, \Gamma_{s_1}$ and $\Lambda_1, \dots, \Lambda_{s_2}$ are row and column vectors, respectively. Thus $\text{ev}_d(f)$ can be viewed as a matrix of size $s_1 \times s_2$. Next we show that $\Gamma_i \in C_{\mathbb{X}_2}(d)$ and $\Lambda_j^\top \in C_{\mathbb{X}_1}(d)$ for all i, j .

Define the following polynomials

$$\begin{aligned}
h_{Q_i} &= f(\alpha_{i,1} \cdot y_1, \alpha_{i,1} \cdot y_2, \dots, \alpha_{i,1} \cdot y_{a_2}, \\
&\quad \alpha_{i,2} \cdot y_1, \alpha_{i,2} \cdot y_2, \dots, \alpha_{i,2} \cdot y_{a_2}, \\
&\quad \vdots \\
&\quad \alpha_{i,a_1} \cdot y_1, \alpha_{i,a_1} \cdot y_2, \dots, \alpha_{i,a_1} \cdot y_{a_2}) \in K[y_1, \dots, y_{a_2}]_d, \text{ and} \\
g_{R_j} &= f(x_1 \cdot \beta_{j,1}, x_1 \cdot \beta_{j,2}, \dots, x_1 \cdot \beta_{j,a_2}, \\
&\quad x_2 \cdot \beta_{j,1}, x_2 \cdot \beta_{j,2}, \dots, x_2 \cdot \beta_{j,a_2}, \\
&\quad \vdots \\
&\quad x_{a_1} \cdot \beta_{j,1}, x_{a_1} \cdot \beta_{j,2}, \dots, x_{a_1} \cdot \beta_{j,a_2}) \in K[x_1, \dots, x_{a_1}]_d.
\end{aligned}$$

Observe that $f(P_{ij}) = h_{Q_i}(R_j) = g_{R_j}(Q_i)$. Noticing the equalities

$$\begin{aligned}
\Gamma_i &= \left(\frac{f(P_{i1})}{f_{i1}(P_{i1})}, \frac{f(P_{i2})}{f_{i2}(P_{i2})}, \dots, \frac{f(P_{is_2})}{f_{is_2}(P_{is_2})} \right) = \\
&\quad \left(\frac{h_{Q_i}(R_1)}{\alpha_{i,k_i}^d \cdot h_1(R_1)}, \frac{h_{Q_i}(R_2)}{\alpha_{i,k_i}^d \cdot h_2(R_2)}, \dots, \frac{h_{Q_i}(R_{s_2})}{\alpha_{i,k_i}^d \cdot h_{s_2}(R_{s_2})} \right) = \frac{1}{(\alpha_{i,k_i})^d} \cdot \text{ev}_d^2(h_{Q_i}), \\
\Lambda_j^\top &= \frac{1}{(\beta_{j,\ell_j})^d} \cdot \text{ev}_d^1(g_{R_j}),
\end{aligned}$$

for $i = 1, \dots, s_1$ and $j = 1, \dots, s_2$, we get that $\Gamma_i \in C_{\mathbb{X}_2}(d)$ and $\Lambda_j^\top \in C_{\mathbb{X}_1}(d)$ for all i, j . This proves that $C_{\mathbb{X}}(d)$ can be regarded as a linear subspace of $C_{\mathbb{X}_1}(d) \otimes C_{\mathbb{X}_2}(d)$. By part (b) and Theorem 2.1 the linear codes $C_{\mathbb{X}}(d)$ and $C_{\mathbb{X}_1}(d) \otimes C_{\mathbb{X}_2}(d)$ have the same dimension. Hence these linear spaces must be equal.

(d): From Theorem 2.1 and part (c), one has $\delta(C_{\mathbb{X}}(d)) = \delta(C_{\mathbb{X}_1}(d))\delta(C_{\mathbb{X}_2}(d))$ for $d \geq 1$.

(e): It follows at once from Theorem 2.2 and part (c).

(f): This follows from Proposition 2.7(ii) and Theorem 2.10(d). \square

Remark 3.2. This result tells us that the direct product of projective Reed-Muller-type codes is again a projective Reed-Muller-type code.

Definition 3.3. If $K^* = K \setminus \{0\}$ and \mathbb{X}_i is the image of $(\mathbb{K}^*)^{a_i}$, under the map $(K^*)^{a_i} \rightarrow \mathbb{P}^{a_i-1}$, $x \rightarrow [x]$, we call \mathbb{X}_i a *projective torus* in \mathbb{P}^{a_i-1} .

Our main theorem gives a wide generalization of most of the main results of [14, 15, 16, 18].

Remark 3.4. If $\mathbb{X}_1 = \mathbb{P}^{a_1-1}$ and $\mathbb{X}_2 = \mathbb{P}^{a_2-1}$, using Theorem 3.1 we recover the formula for the minimum distance of $C_{\mathbb{X}}(d)$ given in [18, Theorem 5.1], and if \mathbb{X}_i is a projective torus for $i = 1, 2$, using Theorem 3.1 we recover the formula for the minimum distance of $C_{\mathbb{X}}(d)$ given in [14, Theorem 5.5]. In these two cases formulas for the basic parameters of $C_{\mathbb{X}_i}(d)$, $i = 1, 2$, are given in [30, Theorem 1] and [28, Theorem 3.5], respectively. We also recover the formulas for the second generalized Hamming weight of some evaluation codes arising from complete bipartite graphs given in [15, Theorem 5.1] and [16, Theorem 3] (see Corollary 3.5).

It turns out that the formula given in Theorem 3.1(e) is a far reaching generalization of the following result.

Corollary 3.5. [15, Theorem 5.1] *Let \mathbb{X} be the Segre product of two projective torus \mathbb{X}_1 and \mathbb{X}_2 . Then the second generalized Hamming weight of $C_{\mathbb{X}}(d)$ is given by*

$$\delta_2(C_{\mathbb{X}}(d)) = \min\{\delta_1(C_{\mathbb{X}_1}(d))\delta_2(C_{\mathbb{X}_2}(d)), \delta_2(C_{\mathbb{X}_1}(d))\delta_1(C_{\mathbb{X}_2}(d))\}.$$

Remark 3.6. The knowledge of the regularity of $K[\mathbf{t}]/I(\mathbb{X})$ is important for applications to coding theory: for $d \geq \text{reg}(K[\mathbf{t}]/I(\mathbb{X}))$ the projective Segre code $C_{\mathbb{X}}(d)$ has minimum distance equal to 1 by Theorem 3.1(f). Thus, potentially good projective Segre codes $C_{\mathbb{X}}(d)$ can occur only if $1 \leq d < \text{reg}(K[\mathbf{t}]/I(\mathbb{X}))$.

Definition 3.7. If \mathbb{X} is parameterized by monomials z^{v_1}, \dots, z^{v_s} , we say that $C_{\mathbb{X}}(d)$ is a *parameterized projective code* of degree d .

Corollary 3.8. *If $C_{\mathbb{X}_i}(d)$ is a parameterized projective code of degree d for $i = 1, 2$, then so is the corresponding projective Segre code $C_{\mathbb{X}}(d)$.*

Proof. It suffices to observe that if \mathbb{X}_1 and \mathbb{X}_2 are parameterized by z^{v_1}, \dots, z^{v_s} and w^{u_1}, \dots, w^{u_r} , respectively, then \mathbb{X} is parameterized by $z^{v_i}w^{u_j}$, $i = 1, \dots, s$, $j = 1, \dots, r$. \square

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